An Overview of Maxwell's Equations

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The four Maxwell's equations are the fundamental basis for electromagnetic theory, and can describe everything about a given electromagnetic field given enough information. Here I will describe each equation, present it in both it's vector and integral forms, and give an example problem one might encounter in a course.

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1 Gauss' Law:

Vector Form:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{1}$$

Integral Form:

$$\oint_S \vec{E} \cdot \mathrm{d}\vec{a} = \frac{Q_{enclosed}}{\epsilon_0} \tag{2}$$

Gauss's law is a way of expressing the divergence of the electric field. In practical terms, this means that if one can pin down either the electric flux through a given closed surface, or the charge enclosed in that surface, one can find the other by this equation. Usually a large amount of symmetry is required in order for this to be an effective method. An interesting result of this equation is that a given amount of charge Q within a given closed surface will cause the same electric flux through that surface regardless of how it is distributed (it could be a point charge, or be spread out evenly, or even as a function of an arbitrary parameter).

1.1 Example

Suppose you have a spherical conductor of radius R, a spherical insulator of radius R and charge density ρ, and a point charge. All three have the same total charge, Q. How do their electric fields compare at 2R from each object. Assume they cannot affect each other. Would the result be the same if we attempted to find the field at some radius less than R?

Due to the symmetry, we can apply Gauss's law, and put in the limits to reflect the spherical geometry being used.

$$\oint_S \vec{E} \cdot \mathrm{d}\vec{a} = \frac{Q_{enclosed}}{\epsilon_0} \tag{3}$$

As the field is not a function of the position on the sphere, we can simply pull it from the integral. Evaluating the integral then yields the surface area of the sphere. In doing so we lose the vector nature of the field, and simply retain it's magnitude. This is acceptable, as it can be surmised to be radially inward or outward depending upon the sign of the charge.

$$E \oint_S \mathrm{d}\vec{a} = \frac{Q_{enclosed}}{\epsilon_0} \tag{4}$$

$$4\pi R^2 E = \frac{Q_{enclosed}}{\epsilon_0} \tag{5}$$

$$E = \frac{Q_{enclosed}}{4\pi R^2 \epsilon_0} \tag{6}$$

At this point, we would normally substitute in the charge enclosed by the closed surface, but here we find that it would be the same value Q. At a radius less than R this result would not hold. The charge on a solid conductor lies on the surface, such that no electric field exists inside a conductor. Inside the insulating sphere the surface would bound some charge, ρ times the smaller enclosed sphere volume, and therefore represent a nonzero electric field. A point charge will continue to look the same regardless of the radius of the surface used.

2 Gauss' Law for Magnetism (no formal name):

Vector Form:

$$\nabla \cdot \vec{B} = 0 \tag{7}$$

Integral Form:

$$\oint_S \vec{B} \cdot \mathrm{d}\vec{a} = 0 \tag{8}$$

Simply put, this law states that there are no magnetic monopoles. For electric charge we can have a single point charge of either positive or negative charge, and have electric field lines that begin or end at infinity, such that a closed surface around the charge will yield a nonzero flux. This is not the case for magnetism, most magnetic dipoles can be represented similarly to a bar magnet, with the field lines running from the north end to the south end, and through the dipole back to the north end. The divergence of any magnetic field must always be zero.

3 Faraday's Law:

Vector Form:

$$\nabla \times \vec{E} = -N\left(\frac{\partial \Phi_B}{\partial t}\right) \tag{9}$$

Integral Form:

$$\oint_L \vec{E} \cdot \mathrm{d}\vec{s} = -N\left(\frac{\partial \Phi_B}{\partial t}\right) \tag{10}$$

If a magnetic field is changing in magnitude or direction, or the area being bounded is somehow changing size or orientation, an induced current may be created. This is the basis for generators that spin a coil of wire in a magnetic field. Even if the field is constant, if we define an area vector normal to the coil, we see that a rotation causes the magnetic flux to change periodically, inducing a current. In vector terms, this is telling us about the curl of the electric field.

3.1 Example

Suppose you have a loop of wire laying in the x-y plane, radius ρ, resistance r, in a uniform magnetic field. The field increases linearly from B_0 tesla in $\hat{z}$ to B_f in $\hat{z}$ in t seconds. Find the induced current in the wire.

Let us begin with the integral form of Faraday's Law. We recognize that N (number of loops) is simply 1, the loop is perpendicular to the field, and uniform across the entire loop. Also of note is that the integral of $\vec{E}\mathrm{d}\vec{s}$ is an alternate definition of voltage.

$$\oint_L \vec{B} \cdot \mathrm{d}\vec{s} = -N\left(\frac{\partial \Phi_B}{\partial t}\right) \tag{11}$$

$$V = -\left(\frac{\partial \Phi_B}{\partial t}\right) \tag{12}$$

Now we must decompose the magnetic flux into terms we can deal with: Area, and magnetic field as a function of time.

$$V = -\frac{\mathrm{d}}{\mathrm{d}t}\pi\rho^2 B(t) \tag{13}$$

Ordinarily we would 'apply' the time derivative to the function describing the magnetic field, but in this case the result has been provided. As the change occurs linearly, we can use ΔB and Δt.

$$V = -\pi\rho^2 \frac{\Delta B}{\Delta t} = -\pi\rho^2 \frac{B_f - B_0}{t} \tag{14}$$

To find the current from the voltage, we simply apply Ohm's law (which assumes the wire is ohmic).

$$V = IR \tag{15}$$

$$I = \frac{V}{R} \tag{16}$$

$$I = \frac{-\pi\rho^2(B_f - B_0)}{tr} \tag{17}$$

4 Ampere's Law:

Vector Form:

$$\nabla \times \vec{B} = \mu_0 J + \mu_0 \epsilon_0 \left(\frac{\partial E}{\partial t}\right) \tag{18}$$

Integral Form:

$$\oint_L \vec{B} \cdot \mathrm{d}\vec{s} = \mu_0 I_{enclosed} + \mu_0 \epsilon_0 \left(\frac{\partial \Phi_E}{\partial t}\right) \tag{19}$$

Just as a magnetic field can produce a current, a current can produce a magnetic field. However, there is a second term that must be accounted for, the 'displacement current', having units of current. This term is required as a magnetic field also exists between capacitor plates, and in other situations where there is no current, but clearly a changing electric field.

4.1 Example

For an infinite wire of radius 0.0012m, carrying a current of 2 amperes with a uniform current density throughout the wire. Find the magnetic field at $r = 0.01m$ and $r = 0.001m$. Recall that $\mu_0 = 4\pi * 10^{-7}$ Tesla meters per ampere.

First we attempt to create an Amperian loop in the same plane as a cross section of the wire, with a radius of 1cm (0.01m). At this point we recognize that the electric field outside the wire is constant, and therefore the displacement current becomes zero, leaving us with the relation below.

$$\oint_L \vec{B} \cdot \mathrm{d}\vec{s} = \mu_0 I_{enclosed} \tag{20}$$

The current enclosed is easily found, as our loop is larger than the wire, so all 2 amperes are enclosed. The magnetic field is not a function of angle away from the wire, only distance. As our loop is centered on the wire, we can simply reduce the integral to a circumference multiplied by the magnetic field magnitude.

$$(2\pi * 0.01)B = 2\mu_0 \tag{21}$$

$$B = \frac{2\mu_0}{2\pi * 0.01} = 4 * 10^{-5} \tag{22}$$

We recognize that Ampere's law yields the general expression for the magnetic field from an infinite wire at a given radius. For the smaller radius, 0.001m, which lies inside the wire, the answer is only slightly more difficult. This time we must define the current enclosed in terms of the wire radius, and the Amperian loop radius (ρ). We do this by taking the ratio of their areas, as for a uniform current distribution this works.

$$\oint_L \vec{B} \cdot \mathrm{d}\vec{s} = \mu_0 I_{total} \frac{\pi\rho^2}{\pi R^2} \tag{23}$$

$$(2\pi\rho)B = \mu_0 I_{total} \frac{\rho^2}{R^2} \tag{24}$$

$$B = \frac{(0.001)^2 \mu_0}{(0.0012)^2 \pi\rho} \tag{25}$$

$$B = 2.78 * 10^{-4} \tag{26}$$