Derivation of the Green's Functions for the Helmholtz and Wave Equations

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1 3D Helmholtz Equation

A Green's Function for the 3D Helmholtz equation must satisfy

$$\nabla^2 G(\mathbf{r},\mathbf{r}_0) + k^2 G(\mathbf{r},\mathbf{r}_0) = \delta(\mathbf{r},\mathbf{r}_0)$$

By Fourier transforming both sides of this equation, we can show that we may take the Green's function to have the form

$$G(\mathbf{r},\mathbf{r}_0) = g(|\mathbf{r}-\mathbf{r}_0|)$$

and that

$$g(r) = 4\pi \int_0^\infty \frac{\text{sinc}(2r\rho)}{k^2 - 4\pi^2\rho^2}\rho^2 \mathrm{d}\rho$$

First we take the Fourier transform of both sides:

$$\mathcal{F}\left(\nabla^2 G(\mathbf{r},\mathbf{r}_0) + k^2 G(\mathbf{r},\mathbf{r}_0)\right) = e^{2\pi i \mathbf{r_0}\rho} \tag{1}$$

$$(2\pi i\rho)^2 G(\rho,\rho_0) + k^2 G(\rho,\rho_0) = e^{2\pi i \mathbf{r_0}\rho} \tag{2}$$

$$G(\rho,\rho_0)\left[(2\pi i\rho)^2 + k^2\right] = e^{-2\pi i \mathbf{r_0}\rho} \tag{3}$$

$$G(\rho,\rho_0) = \frac{e^{-2\pi i \mathbf{r_0}\rho}}{(2\pi i\rho)^2 + k^2} \tag{4}$$

$$G(\mathbf{r},\mathbf{r}_0) = \int_0^\pi \int_0^{2\pi} \int_0^\infty \frac{e^{-2\pi i \mathbf{r_0}\rho}}{(2\pi i\rho)^2 + k^2} e^{2\pi i r\rho\cos\theta}\rho^2 \sin\theta \mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi \tag{5}$$

$$G(\mathbf{r},\mathbf{r}_0) = 2\pi \int_0^\pi \int_0^\infty \frac{e^{2\pi i|r-r_0|\rho\cos\theta}}{k^2 - 4\pi^2\rho^2}\rho^2 \sin\theta \mathrm{d}\rho\mathrm{d}\theta \tag{6}$$

$$u = -\cos\theta \quad \mathrm{d}u = \sin\theta\mathrm{d}\theta \tag{7}$$

$$G(\mathbf{r},\mathbf{r}_0) = 2\pi \int_{-1}^1 \int_0^\infty \frac{e^{-2\pi i|r-r_0|\rho u}}{k^2 - 4\pi^2\rho^2}\rho^2 \mathrm{d}\rho\mathrm{d}u \tag{8}$$

$$G(\mathbf{r},\mathbf{r}_0) = 2\pi \int_0^\infty \frac{\rho^2}{k^2 - 4\pi^2\rho^2} \int_{-1}^1 e^{-2\pi i|r-r_0|\rho u}\mathrm{d}u\mathrm{d}\rho \tag{9}$$

$$G(\mathbf{r},\mathbf{r}_0) = 2\pi \int_0^\infty \frac{\rho^2}{k^2 - 4\pi^2\rho^2} \left[\frac{1}{-2\pi i|r-r_0|\rho}(e^{-2\pi i|r-r_0|\rho} - e^{2\pi i|r-r_0|\rho})\right]\mathrm{d}\rho \tag{10}$$

$$G(\mathbf{r},\mathbf{r}_0) = 2\pi \int_0^\infty \frac{\rho^2}{k^2 - 4\pi^2\rho^2} \left[\frac{2i\sin(-2\pi|r-r_0|\rho)}{-2\pi i|r-r_0|\rho}\right]\mathrm{d}\rho \tag{11}$$

$$G(\mathbf{r},\mathbf{r}_0) = 2\pi \int_0^\infty \frac{\rho^2}{k^2 - 4\pi^2\rho^2} \left[\frac{\sin(-2\pi|r - r_0|\rho)}{-\pi|r - r_0|\rho} \right] \mathrm{d}\rho \tag{12}$$

$$G(\mathbf{r},\mathbf{r}_0) = 4\pi \int_0^\infty \frac{\rho^2}{k^2 - 4\pi^2\rho^2} \mathrm{sinc}(-2\pi|r - r_0|\rho) \mathrm{d}\rho \tag{13}$$

This means that little g has the expected form:

$$g(r) = 4\pi \int_0^\infty \frac{\mathrm{sinc}(2r\rho)}{k^2 - 4\pi^2\rho^2} \rho^2 \mathrm{d}\rho \tag{14}$$

However, this integral passes through a singularity of the integrand. If we use the Cauchy principal value to deal with this problem, we use contour integration we find that the solution is proportional to a spherical wave from a monochromatic point source.

$$\mathrm{Re}\left[\frac{e^{ikr}}{r}\right]$$

$$g(r) = 4\pi \int_0^\infty \frac{\mathrm{sinc}(2r\rho)}{k^2 - 4\pi^2\rho^2} \rho^2 \mathrm{d}\rho = 4\pi \int_0^\infty \frac{\sin(2\pi r\rho)}{(2\pi r\rho)(k^2 - 4\pi^2\rho^2)} \rho^2 \mathrm{d}\rho \tag{15}$$

$$g(r) = 2 \int_0^\infty \frac{\sin(2\pi r\rho)}{r(k^2 - 4\pi^2\rho^2)} \rho \mathrm{d}\rho \tag{16}$$

We recognize that $\mathrm{sinc}(x)$ is an even function, so we can get the same result by integrating over infinite limits and halving the result.

$$g(r) = \int_{-\infty}^\infty \frac{\sin(2\pi r\rho)}{r(k^2 - 4\pi^2\rho^2)} \rho \mathrm{d}\rho \tag{17}$$

$$g(r) = \mathrm{Im}\left\{ \int_{-\infty}^\infty \frac{e^{2\pi i r\rho}}{r(k^2 - 4\pi^2\rho^2)} \rho \mathrm{d}\rho \right\} \tag{18}$$

Now let us focus on solving the integral, then we will take the imaginary part at the end. We recognize two poles at $\rho = \pm\frac{k}{2\pi}$. We must cleverly re-phrase the denominator in order to clearly cancel with our additional factor.

$$\begin{aligned}
\mathrm{Res}_{k/2\pi} &= \left(\rho - \frac{k}{2\pi}\right) \frac{e^{2\pi i r\rho}}{-4\pi^2 r(\rho - \frac{k}{2\pi})(\rho + \frac{k}{2\pi})} \rho \Bigg|_{\rho = k/(2\pi)} \\
&= \frac{e^{2\pi i r\rho}}{-4\pi^2 r(\rho + \frac{k}{2\pi})} \rho \Bigg|_{\rho = k/(2\pi)} \\
&= \frac{e^{irk}}{-4\pi^2 r(\frac{k}{2\pi} + \frac{k}{2\pi})} \frac{k}{2\pi} \\
&= \frac{e^{irk}}{-4\pi r k} \frac{k}{2\pi} \\
&= \frac{e^{irk}}{-8\pi^2 r}
\end{aligned} \tag{19}$$

$$
\begin{aligned}
\text{Res}_{-k/2\pi} &= \left(\rho + \frac{k}{2\pi}\right) \frac{e^{2\pi i r\rho}}{-4\pi^2 r(\rho - \frac{k}{2\pi})(\rho + \frac{k}{2\pi})}\rho\Bigg|_{\rho=-k/(2\pi)} \\
&= \frac{e^{2\pi i r\rho}}{-4\pi^2 r(\rho - \frac{k}{2\pi})}\rho\Bigg|_{\rho=-k/(2\pi)} \\
&= \frac{e^{-irk}}{-4\pi^2 r(\frac{-k}{2\pi} - \frac{k}{2\pi})}\frac{-k}{2\pi} \\
&= \frac{e^{-irk}}{4\pi r k}\frac{-k}{2\pi} \\
&= \frac{e^{-irk}}{-8\pi^2 r}
\end{aligned}
\tag{20}
$$

Now we can utilize the residues, recognizing that since both lie on the line of integration, they have half the influence:

$$
\int_{-\infty}^{\infty} \frac{e^{2\pi i r\rho}}{r(k^2 - 4\pi^2\rho^2)}\rho \mathrm{d}\rho = 2\pi i \sum_i \text{Residue}_i \tag{21}
$$

$$
2\pi i \sum_i \text{Residue}_i = 2\pi i \left(\frac{1}{2}\frac{e^{irk}}{-8\pi^2 r} + \frac{1}{2}\frac{e^{-irk}}{-8\pi^2 r}\right) = i\left(\frac{e^{irk}}{-8\pi r} + \frac{e^{-irk}}{-8\pi r}\right) \tag{22}
$$

$$
2\pi i \sum_i \text{Residue}_i = \frac{i}{-8\pi r}\left(e^{irk} + e^{-irk}\right) = \frac{i}{-8\pi r}\left(\cos(kr) + i\sin(kr) + \cos(-kr) + i\sin(-kr)\right) \tag{23}
$$

$$
2\pi i \sum_i \text{Residue}_i = \frac{i}{-8\pi r}\left(\cos(kr) + i\sin(kr) + \cos(kr) - i\sin(kr)\right) \tag{24}
$$

$$
2\pi i \sum_i \text{Residue}_i = \frac{i}{-8\pi r}\left(\cos(kr) + \cos(kr)\right) = \frac{i\cos(kr)}{-4\pi r} \tag{25}
$$

Finally, we take the imaginary part in order to solve our original integral:

$$
g(r) = 4\pi \int_0^{\infty} \frac{\text{sinc}(2r\rho)}{k^2 - 4\pi^2\rho^2}\rho^2 \mathrm{d}\rho = \frac{\cos(kr)}{-4\pi r} \tag{26}
$$

$$
\boxed{g(r) = \frac{\cos(kr)}{-4\pi r}} \tag{27}
$$

If we go ahead and plot the result of our operator acting upon this function, we do indeed find a delta function!

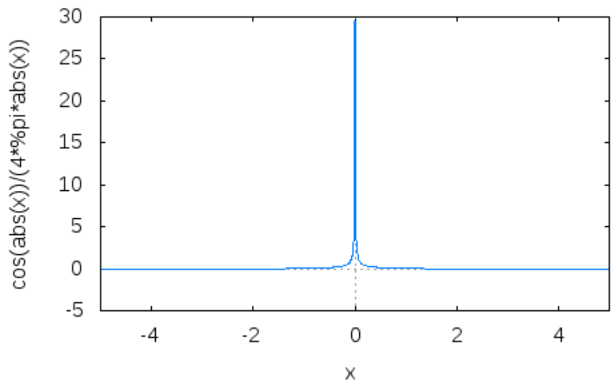
2 3D Wave Equation

A Green's function for the 3D wave equation must satisfy

$$
\nabla^2 G(\mathbf{r}, t, \mathbf{r}_0, t_0) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}G(\mathbf{r}, t, \mathbf{r}_0, t_0) = \delta(\mathbf{r}, \mathbf{r}_0)\delta(t - t_0)
$$

We utilize the space-time Fourier transform, defined as:

$$
F(\rho, \omega) = \int_{\mathbb{R}^3}\int_{-\infty}^{\infty} f(\mathbf{r}, t)e^{-2\pi i(\mathbf{r}\cdot\rho - \omega t)}\mathrm{d}t\mathrm{d}^3 r
$$



Using this Fourier transform we can show that we may assume the Greens' function has the form:

$$G(\mathbf{r}, t, \mathbf{r}_0, t_0) = g(|\mathbf{r} - \mathbf{r}_0|, t - t_0)$$

Instead of performing the whole Fourier transform at once, we instead perform just the time-dependent transform first.

$$\mathcal{F}_t \left(\nabla^2 G(\mathbf{r}, t, \mathbf{r}_0, t_0) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} G(\mathbf{r}, t, \mathbf{r}_0, t_0) \right) = \delta(\mathbf{r}, \mathbf{r}_0) e^{2\pi i t_0 \omega} \tag{28}$$

$$\nabla^2 G(\mathbf{r}, t, \mathbf{r}_0, t_0) - \frac{1}{c^2}(-4\pi^2 t_0^2) G(\mathbf{r}, t, \mathbf{r}_0, t_0) = \delta(\mathbf{r}, \mathbf{r}_0) e^{2\pi i t_0 \omega} \tag{29}$$

$$\left(\nabla^2 + \frac{4\pi^2 \omega^2}{c^2} \right) G(\mathbf{r}, t, \mathbf{r}_0, t_0) = \delta(\mathbf{r}, \mathbf{r}_0) e^{2\pi i t_0 \omega} \tag{30}$$

If we make the substitution $k = \frac{2\pi\omega}{c}$ we see that the wave equation is very similar to the Helmholtz equation we worked with in the previous section.

$$\left(\nabla^2 + k^2 \right) G(\mathbf{r}, t, \mathbf{r}_0, t_0) = \delta(\mathbf{r}, \mathbf{r}_0) e^{2\pi i (t - t_0) \omega} \tag{31}$$

Because of this similarity, we can utilize the solution all the way up to the point where we are inverse-transforming through the ω variable in the next portion of the question. We can use contour integration to show that $g(r, t)$ is a linear combination of an incoming wave

$$\frac{c}{r} \delta(r + ct)$$

and an outgoing wave

$$\frac{c}{r} \delta(r - ct)$$

$$g(r) = 4\pi \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\operatorname{sinc}(2r\rho)}{k^2 - 4\pi^2 \rho^2} \rho^2 e^{2\pi i t_0 \omega} \mathrm{d}\rho \mathrm{d}\omega \tag{32}$$

$$g(r) = \int_{-\infty}^{\infty} \frac{-\cos(kr)}{4\pi r} e^{2\pi i t_0 \omega} \mathrm{d}\omega \tag{33}$$

$$g(r) = \int_{-\infty}^{\infty} \frac{-\cos(kr)}{4\pi r} e^{2\pi i t_0 \omega} \mathrm{d}\omega \tag{34}$$

$$g(r) = \frac{-1}{4\pi r} \int_{-\infty}^{\infty} \frac{1}{2} \left(e^{ikr} + e^{-ikr} \right) e^{2\pi i t_0 \omega} \mathrm{d}\omega \tag{35}$$

$$g(r) = \frac{-1}{8\pi r}\left(\int_{-\infty}^{\infty} e^{ikr}e^{2\pi i t_0 \omega}\mathrm{d}\omega + \int_{-\infty}^{\infty} e^{-ikr}e^{2\pi i t_0 \omega}\mathrm{d}\omega\right) \tag{36}$$

$$g(r) = \frac{-1}{8\pi r}\left(\int_{-\infty}^{\infty} e^{\frac{2\pi i \omega r}{c}}e^{2\pi i t_0 \omega}\mathrm{d}\omega + \int_{-\infty}^{\infty} e^{\frac{-2\pi i \omega r}{c}}e^{2\pi i t_0 \omega}\mathrm{d}\omega\right) \tag{37}$$

$$g(r) = \frac{-1}{8\pi r}\left(\int_{-\infty}^{\infty} e^{2\pi i \omega(\frac{r}{c}+t_0)}\mathrm{d}\omega + \int_{-\infty}^{\infty} e^{-2\pi i \omega(\frac{r}{c}-t_0)}\mathrm{d}\omega\right) \tag{38}$$

$$g(r) = \frac{-1}{4\pi r}\left[\delta\left(\frac{r}{c}+t_0\right) + \delta\left(\frac{r}{c}-t_0\right)\right] \tag{39}$$

$$G(r) = \frac{-1}{4\pi r}\left[\delta\left(\frac{|\mathbf{r}-\mathbf{r}_0|}{c}+(t-t_0)\right) + \delta\left(\frac{|\mathbf{r}-\mathbf{r}_0|}{c}-(t-t_0)\right)\right] \tag{40}$$

To get this to resemble the form alluded to in the question, we must multiply by a form of one, and following the scaling rules for delta functions.

$$G(r) = \frac{-c}{4\pi r}\left[\delta\left(|\mathbf{r}-\mathbf{r}_0| + c(t-t_0)\right) + \delta\left(|\mathbf{r}-\mathbf{r}_0| - c(t-t_0)\right)\right] \tag{41}$$

These are called incoming and outgoing waves because the "location" of the delta function either advances outward from the origin with time (corresponding to the $r - ct$ argument) or shrinks in toward the origin as time grows (corresponding to the $r + ct$ argument). Also a delta function of the radial coordinate looks like a "shell", so the name fits quite well.