An Overview of Third-Order Optical Aberrations

Alexander Miles
AlexanderAshtonMiles@gmail.com

Written: December 29, 2008
Last edit: January 19, 2009

In any optical system, there are aberrations that cause the image produced to be of less than ideal quality. Geometrically, a third-order aberration is one that becomes apparent when we utilize the third order Taylor expansion of sine for Snell's Law. In practice, if one can eliminate (or minimize) the third-order aberrations in their design, the higher order terms and corresponding aberrations can be safely neglected. There are five third order aberrations: spherical, coma, field curvature, astigmatism, and distortion. While not strictly a third order aberration, chromatic aberration will also be described.

---

1 Spherical Aberration

Spherical aberration occurs due to the fact that each ray entering the lens at a different height is focused at a slightly different location. If one were to trace through a paraxial ray (very close to the optical axis), and a ray significantly closer to the edge of the lens, you'd find that the second ray would cross the optical axis (be in focus) much closer to the rear vertex of the lens than the paraxial ray. This aberration occurs in both the longitudinal direction (up and down the optical axis, or z-axis) and the transverse direction (in the ± x direction, being into or out of the page on a normal diagram). These are abbreviated LSA and TSA accordingly. Third order spherical aberration is governed by the following equations, where n is the index of the glass. Note that u is the marginal ray angle.

$$LSA = \frac{r_c^2}{8f}\left[\frac{1}{n^2-n}\right]^2\left[\frac{n+2}{n-1}S^2+(4n+4)Sp+(3n+2)(n-1)^2p^2+\frac{n^3}{n-1}\right] \tag{1}$$

$$\text{TSA} = 2\text{LSA}\tan u \tag{2}$$

In order to minimize spherical aberration, several things can be done. In general, higher index glass will cause less spherical aberration. Due to the fact that very high index glass becomes expensive quickly, the preferred method for minimizing spherical aberration is by the shape factor (S). Shape factor is defined below in terms of radii (R) and surface curvatures (C).

$$S = \frac{R_2+R_1}{R_2-R_1} = \frac{C_1+C_2}{C_1-C_2} \tag{3}$$

The ideal shape factor to minimize longitudinal spherical aberration is given by the relation below, where n is the index of the glass. Position factor (P) must defined, and that follows, where z and z' are the object distance to the front principal plane, and image distance from the back principal plane respectively.

$$S_{LSA} = \frac{-2n^2+2}{n+2}P \tag{4}$$

Note that for an object at or near infinity, the position factor nears -1. For standard glass with index of refraction of about 1.5, the ideal shape factor for spherical aberration is about 0.74.

2 Coma

Coma refers to an aberration where off-axis light is directionally blurred. In practice, it appears like small cones pointing inward or outward from the center of the image. Unlike spherical aberration, coma can be eliminated entirely with the correct shape factor. In addition to shape factor in a singlet, coma can be reduced by using a doublet, and having your design be symmetric about the aperture stop of the system. Coma is governed by the equations below.

$$CMA_3 = \left[\frac{6n+3}{4n}p+\frac{3n+3}{4n^2-4n}S\right]\frac{\overline{u}r_c^2}{3f} \tag{5}$$

In order to eliminate coma in a singlet, the shape factor must obey the following relation.

$$S_{coma} = \frac{-2n^2+n+1}{n+1}P \tag{6}$$

For standard glass with index of refraction of about 1.5, the ideal shape factor for eliminating coma is about 0.8. This is very close to the shape factor for reducing spherical aberration, however few systems have only a singlet lens. In larger systems, the calculations become increasingly complex.

3 Field Curvature

As lenses are spherical, the image plane is not flat, but rather curved. The surface for which the image is in focus is called a Petzval Surface, and it possess a radius given by the following, where n is again the index of the glass, and efl is the effective focal length.

$$R_{img} = n * efl \tag{7}$$

In general a larger radius would yield a flatter image plane, making it easier to image on flat media (film, sensor chips, etc.), therefore higher index glass and longer focal lengths will minimize the effect of field curvature on image quality. It is of note, however, that field flattening lenses do exist, and if the system is sufficiently large, often provide and easy fix.

4 Astigmatism

Astigmatism refers to a system wherein the image is in longitudinal focus at one point in space, but in transverse focus at a different point. Imagine imaging a series of circles with radial lines, and seeing only lines at one position, and only circles at another. While clearly an issue in asymmetric systems, it exists even in systems with circular symmetry about the optical axis.

5 Distortion

Distortion can be recognized by the 'bending' of the image created by a system. If it is pinched inward, it is referred to as 'pincushion distortion', and if it is bowed outward it is referred to as 'barrel distortion'. This is easily remedied by placing the system's aperture stop between the lens elements, rather than in image space (usually causing pincushion distortion) or object space (causing barrel distortion). In the case of a singlet, attempt to ensure that the lens itself is the stop. An air spaced doublet with a stop between the lenses is an easy way to eliminate this aberration.

6 Chromatic Abberration

The index usually quoted for a given glass is often the index for d light, at ###nm wavelength. For other wavelengths, the index is slightly different. A consequence of this is the blue color of the sky during the day, as the blue wavelengths in the suns rays are scattered more effecively than the red. Similarly, at sundown when the sun light passes through the greatest amount of atmosphere before reaching our eyes, we see mostly red, as the blue light has been scattered. In a single lens, each color will come to focus at a slightly different spot in image space. This is not strictly a third order aberration, as even first order treatment of an optical system includes the index of refraction.

Refered for F, d, and C light, we can quantify the chromatic aberration in terms of $\Delta\phi_{FC}$, the difference in power between red and blue wavelengths. We can quickly find that it simplifies down.

$$\Delta\phi_{FC} = (n_F - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) - (n_C - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{8}$$

$$\Delta\phi_{FC} = (n_F - n_C)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{9}$$

Multiply both top and bottom by $(n_d - 1)$ to get it into a more workable form, recognizing the definition of the Abbe number.

$$\Delta\phi_{FC} = \frac{(n_F - n_C)}{(n_d - 1)}(n_d - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{10}$$

$$\Delta\phi_{FC} = \frac{\phi_d}{V\#} \tag{11}$$

With this number in mind, you can find the change in focal length between the visible extrema.